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Quantum state transfer from light beams to atomic ensembles

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Abstract. We show the equivalence between an ensemble of two-level atoms driven by a squeezed vacuum field, and a harmonic oscillator coupled to a squeezed field. We give the conditions for optimal squeezing transfer from the field to the atomic ensemble. We show that EPR-type correlations are created between the atomic ensemble and the incoming field.

PACS. 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

1 Introduction

For quantum information processing as well as for high precision measurements in atomic physics, it is highly desirable to be able to engineer the quantum state of either individual atoms or atomic ensembles. In order to realize quantum registers or quantum memories for the information carried by light, one should be able to map the quantum state of light onto a material system. To reduce the quantum projection noise in measurements, one can consider putting the atoms in squeezed atomic states that exhibit reduced fluctuations for the measurement of interest [1–3]. Atomic ensembles have a variety of superposition states, that can be manipulated by the interaction with light fields. In the case of large enough ensembles, the components of the total polarization of the system, or of its equivalent collective spin, can be considered as a continuous variable, in the same way as the quadrature components of a light field. As was shown by other authors previously, one can get squeezed atomic states by having the atomic ensemble interact with a squeezed field [4–6]. The interaction of atoms with a squeezed field has been a subject of interest for a long time. It has been shown to cause changes in the spectral line shape [7–9]. Here we will concentrate on the generation of squeezed atomic states. We show that this operation can be treated completely analytically while keeping a full quantum treatment if the incoming intensity field is extremely weak. Moreover we show that the resulting atomic state can be correlated at a quantum level with the light input state, leading to EPR-type correlations.

We consider a model system made of an ensemble of 2-level atoms in an optical cavity interacting with a very weak squeezed field. We show that if the mean value of the field is zero, and if the photon number in the field is small *i.e.* for moderate values of the field squeezing, the quantum system is fully equivalent to two coupled harmonic oscillators, an atomic one and a field one. The quantum Langevin equations, taking into account all the noise sources can then be solved in a simple analytical way.

We derive the conditions for obtaining optimal squeezing transfer from a field to an atomic ensemble and we calculate the correlations between the atomic state and the field input state.

2 Model for atomic fluctuations

We consider a set of two level atoms placed inside a single-ended optical cavity and driven by a field the frequency of which is ω_L. The intra-cavity field is represented in the rotating frame by the two dimensionless operators $A(t)$ and $A(t)^\dagger$: $A(t)A(t)^\dagger$ is the number of photons in the cavity at time t. The round-trip time in the cavity is τ, the amplitude transmission coefficient of the coupling mirror is t_{cav}, its amplitude reflection coefficient is r_{cav}, with $r_{cav}^2 + t_{cav}^2 = 1$. The cavity is assumed to have a high finesse ($t_{cav} \ll 1$). The decay rate of the field in the cavity is $\kappa_a = (1 - r_{cav})/\tau = T/2\tau$, where $T = t_{cav}^2$. Let ω_C be the frequency of the cavity resonance which is the closest to ω_L. We define the cavity detuning parameters $\Delta_C = \omega_C - \omega_L$ and $\phi_C = \Delta_C/\kappa_a$.

The atomic frequency is equal to ω_0. We define the atomic detunings $\Delta = \omega_0 - \omega_L$ and $\delta = \Delta/\kappa_a$. The atom-field coupling constant is $g_{at} = \mathcal{E}_0 d/\hbar$, where d is the atomic dipole, and $\mathcal{E}_0 = \sqrt{\hbar\omega_L/2\epsilon_0 \mathcal{S} c\tau}$ corresponds to the electric field of one photon in the cavity mode. In this formula S is the section of the beam. We call γ the atomic dipole decay rate.

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We define the collective polarization $P(t)$ and the collective population difference $S_z(t)$ as:

$$P(t)=\sum_{i=1}^{N} S_i(t) \tag{1}$$

$$P^\dagger(t)=\sum_{i=1}^{N} S_i^\dagger(t) \tag{2}$$

$$S_z(t)=\sum_{i=1}^{N} S_{z\ i}(t) \tag{3}$$

where $S_i(t)$ and $S_i^\dagger(t)$ are the lowering and raising operators for individual atoms in the rotating frame

$$S_i(t)=|g_i\rangle\langle e_i|\,\mathrm{e}^{+\mathrm{i}\omega_\mathrm{L}t} \tag{4}$$

$$S_i^\dagger(t)=|e_i\rangle\langle g_i|\,\mathrm{e}^{-\mathrm{i}\omega_\mathrm{L}t} \tag{5}$$

and $S_{z\ i}(t)$ is given by

$$S_{z\ i}(t)=\frac{1}{2}\left(|e_i\rangle\langle e_i|-|g_i\rangle\langle g_i|\right). \tag{6}$$

The field inside the cavity is related to the incident field A_in and to the atomic polarization by:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t}=-(\kappa_\mathrm{a}+\mathrm{i}\Delta_\mathrm{C})\ A(t)+\mathrm{i}g_\mathrm{at}\ P(t)+\sqrt{2\kappa_\mathrm{a}}\ A_\mathrm{in}(t). \tag{7}$$

The evolution of $A^\dagger(t)$ is given by an equation which is Hermitian conjugate of equation (7). Equation (7) gives the derivative of the intracavity field as coming from the recycling of the field of the cavity, from the field emitted by the atomic polarization and from the incoming field transmitted through the coupling mirror. The fluctuations of the incoming field can be seen as a Langevin force for the intracavity field. The evolution of the atomic polarization and of the populations are given by quantum Langevin equations, derived from the Bloch equations by adding the Langevin forces corresponding to the coupling with the vacuum field surrounding the system

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t}=-(\gamma+\mathrm{i}\Delta)P(t)-2\mathrm{i}g_\mathrm{at}A(t)S_z(t)+F_P(t) \tag{8}$$

$$\frac{\mathrm{d}P^\dagger(t)}{\mathrm{d}t}=-(\gamma-\mathrm{i}\Delta)P^\dagger(t)+2\mathrm{i}g_\mathrm{at}A^\dagger(t)S_z(t)+F_{P^\dagger}(t) \tag{9}$$

$$\frac{\mathrm{d}S_z(t)}{\mathrm{d}t}=-2\gamma(S_z(t)+N/2)-\mathrm{i}g_\mathrm{at}\left(A^\dagger(t)P(t)-A(t)P^\dagger(t)\right)+F_{S_z}(t). \tag{10}$$

The noise operators $F_P(t)$, $F_{P^\dagger}(t)$ and $F_{S_z}(t)$ are characterized by zero averages and by their correlation functions. The non zero ones are equal to [10]:

$$\langle F_P(t)F_{P^\dagger}(t')\rangle=2\gamma N\delta(t-t') \tag{11}$$

$$\langle F_P(t)F_{S_z}(t')\rangle=2\gamma P_0\delta(t-t') \tag{12}$$

$$\langle F_{S_z}(t)F_{P^\dagger}(t')\rangle=2\gamma P_0^*\delta(t-t') \tag{13}$$

$$\langle F_{S_z}(t)F_{S_z}(t'))\rangle=2\gamma(N/2+s_{z0})\delta(t-t') \tag{14}$$

with the atomic steady state mean values:

$$P_0=\langle P(t)\rangle_\mathrm{st},\quad P_0^*=\left\langle P^\dagger(t)\right\rangle_\mathrm{st},\quad s_{z0}=\langle S_z(t)\rangle_\mathrm{st}. \tag{15}$$

We are interested in the quantum fluctuations of the field operator A and of the atomic operators around their steady states mean values. The associated operators are defined by:

$$\delta A_\mathrm{in}(t)=A_\mathrm{in}(t)-a_\mathrm{in},\ \delta A(t)=A(t)-a_0 \tag{16}$$

$$\delta P(t)=P(t)-P_0,\ \delta P^\dagger(t)=P^\dagger(t)-P_0^*,\quad \delta S_z(t)=S_z(t)-s_{z0} \tag{17}$$

with the two mean field values:

$$a_\mathrm{in}=\langle A_\mathrm{in}(t)\rangle_\mathrm{st},\ a_0=\langle A(t)\rangle_\mathrm{st}. \tag{18}$$

In this paper, we consider that the incoming field is a broadband squeezed vacuum field the mean value of which is equal to zero ($a_\mathrm{in}=0$). This assumption implies that:

$$a_0=0,\ P_0=P_0^*=0. \tag{19}$$

If the photon number in the squeezed field remain small, one also has:

$$s_{z0}=-N/2. \tag{20}$$

To obtain equations for the field and atomic fluctuation operators, we linearize equations (7–10). Using equations (19–20), the equations for the atomic fluctuation operators are very simple. We get:

$$\frac{\mathrm{d}\delta P(t)}{\mathrm{d}t}=-(\gamma+\mathrm{i}\Delta)\delta P(t)+\mathrm{i}Ng_\mathrm{at}\ \delta A(t)\ +F_P(t) \tag{21}$$

$$\frac{\mathrm{d}\delta P^\dagger(t)}{\mathrm{d}t}=-(\gamma-\mathrm{i}\Delta)\delta P^\dagger(t)-\mathrm{i}Ng_\mathrm{at}\ \delta A^\dagger(t)\ +F_{P^\dagger}(t) \tag{22}$$

$$\delta S_z(t)=0. \tag{23}$$

These equations show that the evolution of $(P,P^\dagger)$ and S_z are not coupled any more when $a_\mathrm{in}=0$. Furthermore equation (23) implies that the atomic state is an eigenstate of the operator S_z. This property comes from the fact that the autocorrelation function of the Langevin force $F_{S_z}(t)$ given by equation (14) is equal to zero when $s_{z0}=-N/2$. Consequently the noise spectrum of S_z is equal to zero.

Introducing the quantities $p=P/\sqrt{N}$ and $p^\dagger=P^\dagger/\sqrt{N}$ we obtain the set of equations:

$$\frac{\mathrm{d}\delta A(t)}{\mathrm{d}t}=-(\kappa_\mathrm{a}+\mathrm{i}\Delta_\mathrm{C})\delta A(t)+\mathrm{i}g_\mathrm{at}\sqrt{N}\delta p(t)+\sqrt{2\kappa_\mathrm{a}}\delta A_\mathrm{in}(t) \tag{24}$$

$$\frac{\mathrm{d}\delta A^\dagger(t)}{\mathrm{d}t}=-(\kappa_\mathrm{a}-\mathrm{i}\Delta_\mathrm{C})\delta A^\dagger(t)-\mathrm{i}g_\mathrm{at}\sqrt{N}\delta p^\dagger(t)+\sqrt{2\kappa_\mathrm{a}}\delta A_\mathrm{in}^\dagger(t) \tag{25}$$

$$\frac{\mathrm{d}\delta p(t)}{\mathrm{d}t}=-(\gamma+\mathrm{i}\Delta)\delta p(t)+\mathrm{i}g_\mathrm{at}\sqrt{N}\delta A(t)+F_P(t)/\sqrt{N} \tag{26}$$

$$\frac{\mathrm{d}\delta p^\dagger(t)}{\mathrm{d}t}=-(\gamma-\mathrm{i}\Delta)\delta p^\dagger(t)-\mathrm{i}g_\mathrm{at}\sqrt{N}\delta A^\dagger(t)+F_{P^\dagger}(t)/\sqrt{N}. \tag{27}$$

The atomic fluctuations can be easily deduced from this system of four coupled equations. Before giving the result, we will study the quantum fluctuations of a harmonic oscillator placed in the same conditions and show that the two systems have a similar behavior.

3 Model for harmonic oscillator fluctuations

We consider a harmonic oscillator described by the operators $B(t)$ and $B^\dagger(t)$. This harmonic oscillator is placed inside the cavity defined in the previous section. The free evolution frequency of $B(t)$ is equal to ω_b. The harmonic oscillator is represented in the rotating frame by the operator $b(t) = B(t)\mathrm{e}^{\mathrm{i}\omega_\mathrm{L}t}$ and $b^\dagger(t) = B^\dagger(t)\mathrm{e}^{-\mathrm{i}\omega_\mathrm{L}t}$. We define the detunings $\Delta_\mathrm{b} = \omega_\mathrm{b} - \omega_\mathrm{L}$ and $\phi_\mathrm{b} = \Delta_\mathrm{b}/\kappa_\mathrm{a}$. We call κ_b the decay rate of $b(t)$ and $b^\dagger(t)$, due to the coupling with a reservoir in the vacuum state [11]. The coupling between the harmonic oscillator and the field is given by the Hamiltonian $H_I = \hbar g(A^\dagger b + b^\dagger A)$. The Heisenberg evolution equations of the system operators are:

$$\frac{\mathrm{d}b(t)}{\mathrm{d}t} = -(\kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})b(t) + \mathrm{i}gA(t) + \sqrt{2\kappa_\mathrm{b}}b_\mathrm{in}(t) \quad (28)$$

$$\frac{\mathrm{d}b^\dagger(t)}{\mathrm{d}t} = -(\kappa_\mathrm{b} - \mathrm{i}\Delta_\mathrm{b})b^\dagger(t) - \mathrm{i}gA^\dagger(t) + \sqrt{2\kappa_\mathrm{b}}b^\dagger_\mathrm{in}(t) \quad (29)$$

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = -(\kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})A(t) + \mathrm{i}gb(t) + \sqrt{2\kappa_\mathrm{a}}A_\mathrm{in}(t) \quad (30)$$

$$\frac{\mathrm{d}A^\dagger(t)}{\mathrm{d}t} = -(\kappa_\mathrm{a} - \mathrm{i}\Delta_\mathrm{C})A^\dagger(t) - \mathrm{i}gb^\dagger(t) + \sqrt{2\kappa_\mathrm{a}}A^\dagger_\mathrm{in}(t). \quad (31)$$

In these equations the source terms proportional to $A_\mathrm{in}(t)$, $A^\dagger_\mathrm{in}(t)$ and $b_\mathrm{in}(t)$, $b^\dagger_\mathrm{in}(t)$ correspond to the coupling of the field and of the harmonic oscillator with their respective baths.

We introduce the harmonic oscillator mean values $b = \langle b(t)\rangle_\mathrm{st}$ and $b_\mathrm{in} = \langle b_\mathrm{in}(t)\rangle_\mathrm{st}$. We assume that the mean value of $b_\mathrm{in}(t)$ is equal to zero ($b_\mathrm{in}(t) = \delta b_\mathrm{in}(t)$). The mean values a and b can be easily computed from equations (28–31) without the fluctuating terms.

The steady state of this system is:

$$b = \frac{\mathrm{i}g\sqrt{2\kappa_\mathrm{a}}a_\mathrm{in}}{g^2 + (\kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})(\kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})},$$

$$a = \frac{(\kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})\sqrt{2\kappa_\mathrm{a}}a_\mathrm{in}}{g^2 + (\kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})(\kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})}\cdot \quad (32)$$

The mean value of the reflected field is deduced from the preceding equation by using the input-output relations for the field:

$$a_\mathrm{out} = r_\mathrm{cav}a_\mathrm{in} - t_\mathrm{cav}a \quad (33)$$

and we obtain:

$$a_\mathrm{out} = \frac{-g^2 + (\kappa_\mathrm{a} - \mathrm{i}\Delta_\mathrm{C})(\kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})}{g^2 + (\kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})(\kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})}a_\mathrm{in}. \quad (34)$$

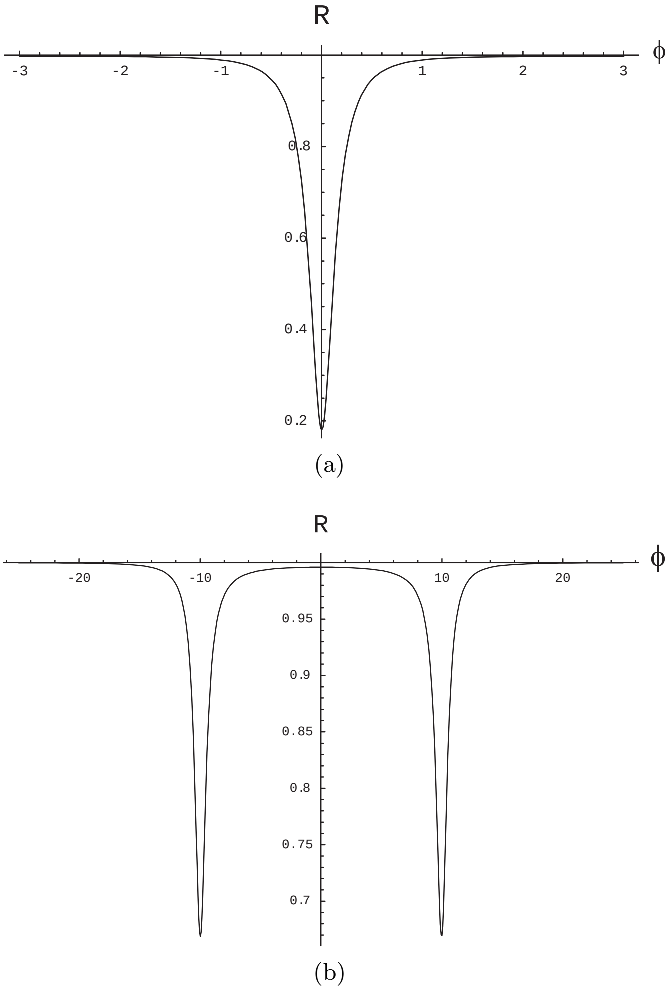

Fig. 1. Reflection coefficient R of a Fabry Perot cavity containing a harmonic oscillator function of the normalized detuning ϕ between field and cavity for $\rho = 0.1$. In (a) $\eta = 0.2$, the coupling between the harmonic oscillator and the field is weak while in (b) $\eta = 10$, corresponding to the strong coupling regime.

Defining the coupling parameter $\eta = g/\kappa_\mathrm{a}$ and the ratio between the dampings $\rho = \kappa_\mathrm{b}/\kappa_\mathrm{a}$, we can rewrite the preceding equation as:

$$R = \left|\frac{a_\mathrm{out}}{a_\mathrm{in}}\right|^2 = \left|\frac{-\eta^2 + (1 - \mathrm{i}\phi_\mathrm{C})(\rho + \mathrm{i}\phi_\mathrm{b})}{\eta^2 + (1 + \mathrm{i}\phi_\mathrm{C})(\rho + \mathrm{i}\phi_\mathrm{b})}\right|^2\cdot \quad (35)$$

The shape of the reflected field intensity as a function of ϕ ($\phi = \phi_\mathrm{C} = \phi_\mathrm{b}$) depends on the value of η, with only one resonance at $\phi = 0$ if the coupling is weak ($\eta < 1, \rho$), or two resonances around $\phi = \pm\sqrt{\eta^2 - (1 + \rho^2)/2}$ if the strong coupling regime [12] is achieved ($\eta > 1, \rho$). Figure 1 gives the variation of R as a function of ϕ with $\rho = 0.1$, for increasing values of η. For $0 < \eta < 1$, the reflected intensity has a dip around $\phi = 0$ (see Fig. 1a). This dip is due to an energy transfer from the field to the harmonic oscillator, and when η increases this hole broadens. For $\eta > 1$ the reflected intensity has two minima, which are obtained for $\phi = \pm\eta$ when $\eta \gg 1$ (see Fig. 1b which corresponds to $\eta = 10$).

Now, we are going to compute the fluctuations of the field and harmonic oscillator operators. We are interested in the quantum fluctuations of the operators $A\,(t)$ and $b\,(t)$ around their steady state mean values. We define:

$$\delta b\,(t) = b\,(t) - b. \tag{36}$$

Equations (28–31) give the following equations for the fluctuations:

$$\frac{\mathrm{d}\delta b(t)}{\mathrm{d}t} = -(\kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})\delta b(t) + \mathrm{i}g\delta A(t) + \sqrt{2\kappa_\mathrm{b}}\delta b_\mathrm{in}(t) \tag{37}$$

$$\frac{\mathrm{d}\delta b^\dagger(t)}{\mathrm{d}t} = -(\kappa_\mathrm{b} - \mathrm{i}\Delta_\mathrm{b})\delta b^\dagger(t) - \mathrm{i}g\delta A^\dagger(t) + \sqrt{2\kappa_\mathrm{b}}\delta b_\mathrm{in}^\dagger(t) \tag{38}$$

$$\frac{\mathrm{d}\delta A(t)}{\mathrm{d}t} = -(\kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})\delta A(t) + \mathrm{i}g\delta b(t) + \sqrt{2\kappa_\mathrm{a}}\delta A_\mathrm{in}(t) \tag{39}$$

$$\frac{\mathrm{d}\delta A^\dagger(t)}{\mathrm{d}t} = -(\kappa_\mathrm{a} - \mathrm{i}\Delta_\mathrm{C})\delta A^\dagger(t) - \mathrm{i}g\delta b^\dagger(t) + \sqrt{2\kappa_\mathrm{a}}\delta A_\mathrm{in}^\dagger(t). \tag{40}$$

Let us notice that these equations are valid even for large values of the mean field and of the photon numbers, on contrast to equations (21–23). We see that equations (24–27) are identical to equations (37–40) with the equivalences:

$$p \leftrightarrow b,\ \ g_\mathrm{at}\sqrt{N} \leftrightarrow g,\ \ \Delta \leftrightarrow \Delta_\mathrm{b}\ \text{ and }\ \delta b_\mathrm{in}(t) \leftrightarrow \frac{F_P(t)}{\sqrt{N}}\cdot \tag{41}$$

Consequently, the same results will be obtained for the variances of the atomic polarization p with the 2-level atom model and for the variances of the annihilation operator b with the harmonic oscillator model.

To solve the system of equations, it is useful to work in Fourier space. For any operator $O(t)$ in the rotating frame we define the Fourier transform $O(\omega)$ as:

$$O(\omega) = \int O(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \tag{42}$$

$$O^\dagger(\omega) = \int O^\dagger(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t. \tag{43}$$

The Fourier components of the fluctuation operators are then given by:

$$(-\mathrm{i}\omega + \kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})\delta b(\omega) = \mathrm{i}g\delta A(\omega) + \sqrt{2\kappa_\mathrm{b}}\delta b_\mathrm{in}(\omega) \tag{44}$$

$$(-\mathrm{i}\omega + \kappa_\mathrm{b} - \mathrm{i}\Delta_\mathrm{b})\delta b^\dagger(\omega) = -\mathrm{i}g\delta A^\dagger(\omega) + \sqrt{2\kappa_\mathrm{b}}\delta b_\mathrm{in}^\dagger(\omega) \tag{45}$$

$$(-\mathrm{i}\omega + \kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})\delta A(\omega) = \mathrm{i}g\delta b(\omega) + \sqrt{2\kappa_\mathrm{a}}\delta A_\mathrm{in}(\omega) \tag{46}$$

$$(-\mathrm{i}\omega + \kappa_\mathrm{a} - \mathrm{i}\Delta_\mathrm{C})\delta A^\dagger(\omega) = -\mathrm{i}g\delta b^\dagger(\omega) + \sqrt{2\kappa_\mathrm{a}}\delta A_\mathrm{in}^\dagger(\omega). \tag{47}$$

This system of coupled equations can easily be solved by introducing the quantities:

$$J_\mathrm{a}(\omega) = \frac{\mathrm{i}g\sqrt{2\kappa_\mathrm{a}}}{g^2 + (-\mathrm{i}\omega + \kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})(-\mathrm{i}\omega + \kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})} \tag{48}$$

$$J_\mathrm{b}(\omega) = \frac{\sqrt{2\kappa_\mathrm{b}}\,(-\mathrm{i}\omega + \kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})}{g^2 + (-\mathrm{i}\omega + \kappa_\mathrm{a} + \mathrm{i}\Delta_\mathrm{C})(-\mathrm{i}\omega + \kappa_\mathrm{b} + \mathrm{i}\Delta_\mathrm{b})}\cdot \tag{49}$$

We obtain:

$$\delta b(\omega) = J_\mathrm{a}(\omega)\delta A_\mathrm{in}(\omega) + J_\mathrm{b}(\omega)\delta b_\mathrm{in}(\omega) \tag{50}$$

$$\delta b^\dagger(\omega) = (J_\mathrm{a}(-\omega))^*\delta A_\mathrm{in}^\dagger(\omega) + (J_\mathrm{b}(-\omega))^*\delta b_\mathrm{in}^\dagger(\omega). \tag{51}$$

Using the parameters η and ρ we can rewrite equations (48, 49) with the normalized frequency $\Omega = \omega/\kappa_\mathrm{a}$ as:

$$J_\mathrm{a}(\Omega) = \frac{\mathrm{i}\eta\sqrt{2}}{\sqrt{\kappa_\mathrm{a}}D(\Omega)} \tag{52}$$

$$J_\mathrm{b}(\Omega) = \frac{\sqrt{2\rho}(1 + \mathrm{i}\phi_\mathrm{C} - \mathrm{i}\Omega)}{\sqrt{\kappa_\mathrm{a}}D(\Omega)} \tag{53}$$

with:

$$D(\Omega) = \eta^2 + (1 + \mathrm{i}\phi_\mathrm{C} - \mathrm{i}\Omega)(\rho + \mathrm{i}\phi_\mathrm{b} - \mathrm{i}\Omega). \tag{54}$$

To calculate the fluctuations we write equations (50, 51) in a matrix form. We introduce the 2-dimensional vectors $|\delta b(\omega)]$, $|\delta b_\mathrm{in}(\omega)]$ and $|\delta A_\mathrm{in}(\omega)]$:

$$|\delta b(\omega)] = \left|\begin{matrix}\delta b(\omega)\\ \delta b^\dagger(\omega)\end{matrix}\right],\ \ |\delta b_\mathrm{in}(\omega)] = \left|\begin{matrix}\delta b_\mathrm{in}(\omega)\\ \delta b_\mathrm{in}^\dagger(\omega)\end{matrix}\right],$$
$$|\delta A_\mathrm{in}(\omega)] = \left|\begin{matrix}\delta A_\mathrm{in}(\omega)\\ \delta A_\mathrm{in}^\dagger(\omega)\end{matrix}\right] \tag{55}$$

and their adjoints (for example $[\delta b(\omega)| = |\delta b(\omega)]^\dagger = \left[\delta b^\dagger(-\omega), \delta b(-\omega)\right|$). Then equations (37–40) can be written under the form:

$$|\delta b(\omega)] = [J_\mathrm{a}(\omega)]\,|\delta A_\mathrm{in}(\omega)] + [J_\mathrm{b}(\omega)]\,|\delta b_\mathrm{in}(\omega)] \tag{56}$$

where the 2×2 matrices $[J_\mathrm{a}(\omega)]$ and $[J_\mathrm{b}(\omega)]$ are respectively equal to:

$$[J_\mathrm{a}(\omega)] = \begin{bmatrix} J_\mathrm{a}(\omega) & 0\\ 0 & J_\mathrm{a}^*(-\omega)\end{bmatrix} \tag{57}$$

and:

$$[J_\mathrm{b}(\omega)] = \begin{bmatrix} J_\mathrm{b}(\omega) & 0\\ 0 & J_\mathrm{b}^*(-\omega)\end{bmatrix}. \tag{58}$$

The covariance matrices $[V_{A_\mathrm{in}}(\omega)]$ of the incoming field and $[V_{b_\mathrm{in}}(\omega)]$ of the harmonic oscillator bath are defined as:

$$\langle|\delta A_\mathrm{in}(\omega)]\,[\delta A_\mathrm{in}(\omega')|\rangle = 2\pi\,[V_{A_\mathrm{in}}(\omega)]\,\delta\,(\omega - \omega') \tag{59}$$

$$\langle|\delta b_\mathrm{in}(\omega)]\,[\delta b_\mathrm{in}(\omega')|\rangle = 2\pi\,[V_{b_\mathrm{in}}(\omega)]\,\delta\,(\omega - \omega')\,. \tag{60}$$

The fluctuations of the harmonic oscillator bath are taken to be at the standard quantum level, and the matrix $[V_{b_{\rm in}}(\omega)]$ is consequently equal to [13]:

$$[V_{b_{\rm in}}(\omega)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{61}$$

For a broadband squeezed field, the field fluctuations are given by [14]:

$$[V_{A_{\rm in}}(\omega)] = \begin{bmatrix} \cosh^2(r) & \sinh(2r)/2 \\ \sinh(2r)/2 & \sinh^2(r) \end{bmatrix}. \tag{62}$$

The quadrature operators of the incoming field are defined as:

$$\hat{E}_{{\rm in},\varphi}(t) = A_{\rm in}(t){\rm e}^{{\rm i}\varphi} + A^\dagger_{\rm in}(t){\rm e}^{-{\rm i}\varphi}. \tag{63}$$

The noise spectrum of $\hat{E}_{{\rm in},\varphi}(t)$ will be denoted $S_{\hat{E}_{{\rm in},\varphi}}(\omega)$:

$$\langle \delta\hat{E}_{{\rm in},\varphi}(\omega)\delta\hat{E}_{{\rm in},\varphi}(\omega')\rangle = 2\pi S_{\hat{E}_{{\rm in},\varphi}}(\omega)\delta(\omega+\omega') \tag{64}$$

and is obtained from the matrix elements of $[V_{A_{\rm in}}(\omega)]$:

$$S_{\hat{E}_{{\rm in},\varphi}}(\omega) = V_{A_{\rm in}1,1}(\omega) + V_{A_{\rm in}2,2}(\omega) + 2\,{\rm Re}({\rm e}^{2{\rm i}\varphi}V_{A_{\rm in}1,2}(\omega)). \tag{65}$$

In the case of the broadband squeezed field equations (62, 65) show that the quadrature with the minimal noise spectrum is obtained for $\varphi = \pi/2$:

$$S_{\hat{E}_{{\rm in},\varphi=\frac{\pi}{2}}}(\omega) = {\rm e}^{-2r}. \tag{66}$$

The parameter r gives the amount of squeezing, equal to ${\rm e}^{-2r}$. The quadrature corresponding to $\varphi = 0$ is the noisiest one:

$$S_{\hat{E}_{{\rm in},\varphi=0}}(\omega) = {\rm e}^{2r}. \tag{67}$$

When $r = 0$ the fluctuations of the standard quantum limit are recovered.

We define the covariance matrix $[S_{\rm b}(\omega)]$ of the harmonic oscillator as:

$$\langle |\delta b(\omega)]\,[\delta b(\omega')|\rangle = 2\pi\,[S_{\rm b}(\omega)]\,\delta(\omega-\omega'). \tag{68}$$

Using equations (56, 59, 60), we obtain the following expression for $[S_{\rm b}(\omega)]$:

$$[S_{\rm b}(\omega)] = [J_{\rm a}(\omega)]\,[V_{A_{\rm in}}(\omega)]\,[J_{\rm a}(\omega)]_{\rm hc} + [J_{\rm b}(\omega)]\,[V_{b_{\rm in}}(\omega)]\,[J_{\rm b}(\omega)]_{\rm hc} \tag{69}$$

where $[J_{\rm a}(\omega)]_{\rm hc}$ (resp. $[J_{\rm b}(\omega)]_{\rm hc}$) is the Hermitian conjugate matrix of $[J_{\rm a}(\omega)]$ (resp. $[J_{\rm b}(\omega)]$).

Using equations (61, 62), and the expressions we have obtained for $[J_{\rm a}(\omega)]_{\rm hc}$ and $[J_{\rm b}(\omega)]_{\rm hc}$, we can rewrite equation (69) as:

$$[S_{\rm b}(\omega)] = \begin{bmatrix} \cosh^2(r)\,|J_{\rm a}(\omega)|^2 + |J_{\rm b}(\omega)|^2 & J_{\rm a}(\omega)J_{\rm a}(-\omega)\sinh(2r)/2 \\ J^*_{\rm a}(\omega)J^*_{\rm a}(-\omega)\sinh(2r)/2 & \sinh^2(r)\,|J_{\rm a}(-\omega)|^2 \end{bmatrix}. \tag{70}$$

The quadrature operators of the harmonic oscillator are defined by:

$$Y_\varphi(t) = b(t){\rm e}^{{\rm i}\varphi} + b^\dagger(t){\rm e}^{-{\rm i}\varphi}. \tag{71}$$

The corresponding noise spectrum $S_{Y_\varphi}(\omega)$ is calculated using equation (68), and can then be evaluated with the matrix elements of $[S_{\rm b}(\omega)]$:

$$S_{Y_\varphi}(\omega) = S_{{\rm b}1,1}(\omega) + S_{{\rm b}2,2}(\omega) + 2\,{\rm Re}({\rm e}^{2{\rm i}\varphi}S_{{\rm b}1,2}(\omega)). \tag{72}$$

Using equation (70), we obtain:

$$S_{Y_\varphi}(\omega) = \cosh^2(r)\,|J_{\rm a}(\omega)|^2 + \sinh^2(r)\,|J_{\rm a}(-\omega)|^2 + |J_{\rm b}(\omega)|^2 + {\rm Re}\{{\rm e}^{2{\rm i}\varphi}\sinh(2r)J_{\rm a}(\omega)J_{\rm a}(-\omega)\}. \tag{73}$$

When the incoming field is in a coherent state ($r = 0$), the noise spectra of the quadratures of the harmonic oscillator do not depend on φ and are equal to:

$$S_{Y_\varphi}(\omega)_{\rm coherent} = |J_{\rm a}(\omega)|^2 + |J_{\rm b}(\omega)|^2. \tag{74}$$

The variance of any operator O is defined by $\Delta O = \langle \delta O^2\rangle$. Then the variance ΔY_φ of the φ quadrature of the harmonic oscillator is given by:

$$\Delta Y_\varphi = \frac{1}{2\pi}\int {\rm d}\omega\, S_{Y_\varphi}(\omega). \tag{75}$$

Squeezing is obtained when one of the quadrature operators has a variance smaller than one. Considering the minimal variance $\Delta Y_{\rm min}$, the condition for having squeezing is then $\Delta Y_{\rm min} < 1$.

Using equations (73, 75), we find the value of $\Delta Y_{\rm min}$ when the incoming field is a broadband squeezed field:

$$\Delta Y_{\rm min} = I_{\rm b} + \cosh(2r)I_{\rm a} - \sinh(2r)I_{\rm c}. \tag{76}$$

We have introduced the following integrals:

$$I_{\rm a} = \frac{1}{2\pi}\int {\rm d}\omega\,|J_{\rm a}(\omega)|^2 \tag{77}$$

$$I_{\rm b} = \frac{1}{2\pi}\int {\rm d}\omega\,|J_{\rm b}(\omega)|^2 \tag{78}$$

$$I_{\rm c} = \frac{1}{2\pi}\left|\int {\rm d}\omega\, J_{\rm a}(\omega)J_{\rm a}(-\omega)\right|. \tag{79}$$

When the incoming field is in a coherent state ($r = 0$), the harmonic oscillator is also in a coherent state, and the variances of all the quadrature components are equal to one, as in the case of the ground state. This property implies that:

$$I_{\rm a} + I_{\rm b} = 1. \tag{80}$$

When the incoming field is squeezed, the harmonic oscillator can be squeezed as well if the integral $I_{\rm c}$ is large enough. $I_{\rm c}$ satisfies the following inequalities:

$$I_{\rm c} \le \frac{1}{2\pi}\int {\rm d}\omega\,|J_{\rm a}(\omega)J_{\rm a}(-\omega)| \le \frac{1}{2\pi}\int {\rm d}\omega\,|J_{\rm a}(\omega)|^2 = I_{\rm a} \tag{81}$$

and consequently:

$$\Delta Y_{\min} \geq I_{\rm b} + (\cosh(2r) - \sinh(2r))I_{\rm a} = I_{\rm b} + I_{\rm a}{\rm e}^{-2r}. \quad (82)$$

The equality between $I_{\rm c}$ and $I_{\rm a}$ is obtained when the argument of $(J_{\rm a}(\omega)J_{\rm a}(-\omega))$ is independent of ω, and $|J_{\rm a}(\omega)| = |J_{\rm a}(-\omega)|$. This condition is verified if:

$$\Delta_{\rm C} = \Delta_{\rm b} = 0 \quad (83)$$

or if:

$$\Delta_{\rm C} = -\Delta_{\rm b},\ \kappa_{\rm a} = \kappa_{\rm b}. \quad (84)$$

The second condition, given by (84), is of minor interest since it corresponds to a squeezing transfer from the field to the atoms limited to 50% as we will see below. In the following we will only consider the first condition, given by (83). When (83) is fulfilled, $J_{\rm a}(-\omega) = -J_{\rm a}(\omega)^*$, and (73) becomes:

$$S_{Y_\varphi}(\omega)_{\Delta_{\rm C}=\Delta_{\rm b}=0} = |J_{\rm b}(\omega)|^2 + |J_{\rm a}(\omega)|^2 (\cosh(2r) - \sinh(2r)\cos(2\varphi)). \quad (85)$$

The quadrature component corresponding to $\varphi = 0$ has the minimal noise spectrum for all the values of the noise frequency:

$$S_{Y\varphi=0}(\omega)_{(\Delta_{\rm C}=\Delta_{\rm b}=0)} = {\rm e}^{-2r}|J_{\rm a}(\omega)|^2 + |J_{\rm b}(\omega)|^2 \quad (86)$$

and consequently it has the minimal variance:

$$\Delta Y_{\min\ (\Delta_{\rm C}=\Delta_{\rm b}=0)} = \Delta Y_{\varphi=0} = I_{\rm b} + I_{\rm a}{\rm e}^{-2r} \geqslant {\rm e}^{-2r}. \quad (87)$$

The latter inequality is due to equation (80). It means that the squeezing of the harmonic oscillator is smaller than the squeezing of the field, due to its coupling with a reservoir. Then equations (83, 84) can be seen as optimal squeezing transfer conditions from field to the harmonic oscillator.

When one of this squeezing transfer conditions is verified, the integrals $I_{\rm b}$ and $I_{\rm a}$ can be easily calculated, and in the case of equation (83):

$$I_{a(\Delta_{\rm C}=\Delta_{\rm b}=0)} = \eta^2 \frac{1}{1+\rho}\frac{1}{\eta^2+\rho} \quad (88)$$

$$I_{b(\Delta_{\rm C}=\Delta_{\rm b}=0)} = \frac{\rho}{1+\rho}\left(1 + \frac{1}{\eta^2+\rho}\right). \quad (89)$$

Consequently the value of $\Delta Y_{\min}$ is equal to:

$$\Delta Y_{\min\ (\Delta_{\rm C}=\Delta_{\rm b}=0)} = 1 + \left({\rm e}^{-2r} - 1\right)\frac{\eta^2}{(1+\rho)(\eta^2+\rho)}\cdot \quad (90)$$

To have $\Delta Y_{\min}$ as small as possible, one needs $\eta^2 \gg \rho$. Then (90) writes:

$$\Delta Y_{\min\ (\Delta_{\rm C}=\Delta_{\rm b}=0,\,\eta^2\gg\rho)} = 1 + \frac{\left({\rm e}^{-2r} - 1\right)}{1+\rho}\cdot \quad (91)$$

For a perfectly squeezed field ($r \to \infty$), we get:

$$\Delta Y_{\min\ (\Delta_{\rm C}=\Delta_{\rm b}=0,\,r\to\infty,\,\eta^2\gg\rho)} = \frac{\rho}{1+\rho} = \frac{\kappa_{\rm b}}{\kappa_{\rm b}+\kappa_{\rm a}}\cdot \quad (92)$$

The best squeezing transfer then corresponds to $\rho \ll 1$, which means that the relaxation rate of the harmonic oscillator, $\kappa_{\rm b}$, is much smaller than that of the cavity, $\kappa_{\rm a}$. We would obtain the same expression for $\Delta_{\rm C} = -\Delta_{\rm b}$, $\kappa_{\rm a} = \kappa_{\rm b}$, $\eta^2 \gg \rho$ which shows that in this case the squeezing of the harmonic oscillator is limited to 50%.

4 Spin squeezing

In the same way as a squeezed state of the electromagnetic field is defined by comparison to the coherent state, a squeezed spin state will be defined as having fluctuations in one component lower than the one of a coherent spin state [3]. Since the noise spectrum of a coherent spin state is not white, contrary to the one of a freely propagating coherent light field, one has to compare the variances of the considered spin components to the variances of the components of a coherent spin state. We introduce S_x and S_y:

$$S_x = \frac{P+P^\dagger}{2},\ S_y = \frac{P-P^\dagger}{2{\rm i}}\cdot \quad (93)$$

Due to the commutation relations of an angular momentum $[S_j, S_k] = {\rm i}S_l$, where $j, k, l = x, y, z$, the variances of two orthogonal components S_u and S_v of the spin in the x, y-plane obey a Heisenberg inequality:

$$\sqrt{\Delta S_u \Delta S_v} \leq \frac{|\langle S_z\rangle|}{2}\cdot \quad (94)$$

Since the field has a zero mean value, the mean value of the spin is aligned with the z-axis, and consequently there is spin squeezing if one of the spin components in the (x, y)-plane has a variance smaller than the reference given by the Heisenberg inequality ($|\langle S_z\rangle|/2$) [3,2], which is equal to $N/4$ according to equation (19). We call $\Delta S_{\min}$ the minimal variance in the (x, y)-plane normalized to $N/4$ which is found for some direction $\mathbf{u}$ and we compare its value with 1.

In order to evaluate $\Delta S_{\min}$ in the case of interaction between atoms and zero mean field value, we use the equivalence given by equation (41). We see that we will obtain the same value for the spin components in the (x, y)-plane as the one we obtained for the quadrature components of the harmonic oscillator in Section 2. In particular, we obtain the equality:

$$\Delta S_{\min} = \Delta Y_{\min}. \quad (95)$$

As mentioned above this equivalence is valid when the photon number in the squeezed incoming field is small. The model also allows to show that the interaction of the spin with a squeezed field also causes changes in the spectral line shape of the atoms. This effect was studied in detail in [9,15], for the case of the bad cavity limit.

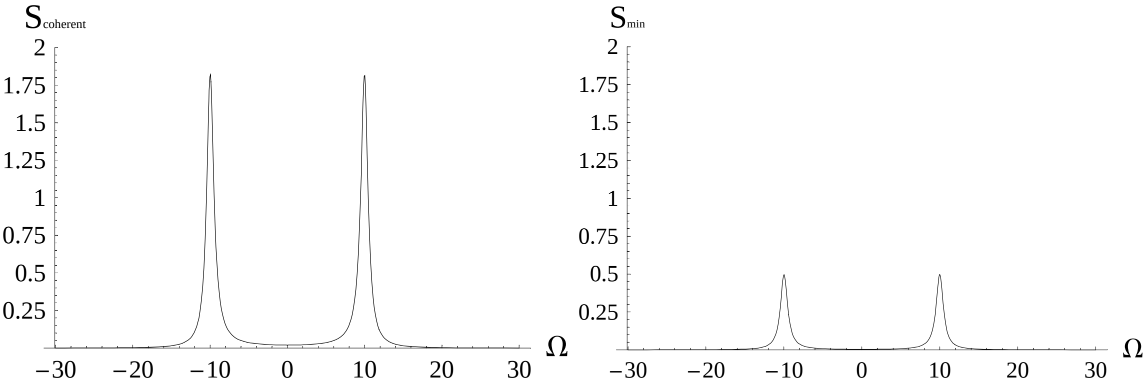

Fig. 2. Noise spectra of harmonic oscillator quadratures *versus* the normalized frequency Ω in the conditions of good squeezing transfer for $\rho = 0.1$, $\eta = 10$. In (a) the incoming field is in a coherent state, and all the quadrature have the same spectra $S_{\text{coherent}}(\Omega)$. In (b) the amount of squeezing of the incoming field is 80% and $S_{\min}(\Omega)$ the spectrum of the squeezed quadrature component is plotted.

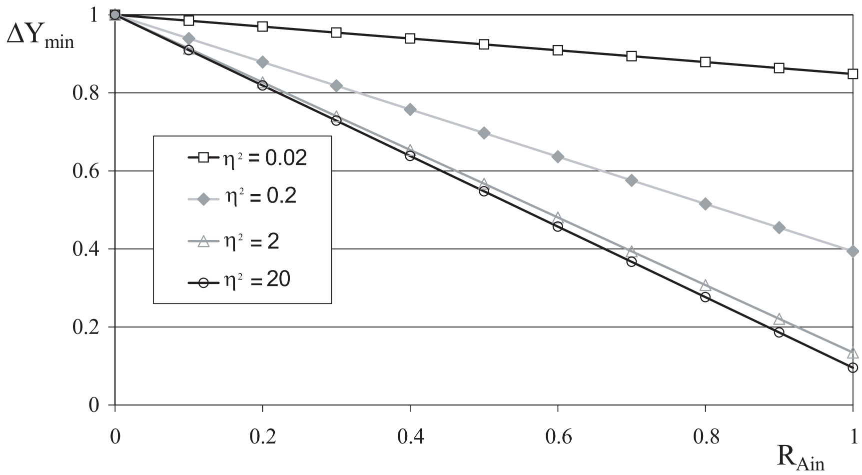

Fig. 3. Evolution of the minimal variance $\Delta Y_{\min}$ of the quadrature of the harmonic oscillator with the squeezing rate of the incoming field $R_{A_{\text{in}}}$ for different values of the coupling constant η in the optimal squeezing transfer regime.

In order to minimize $\Delta S_{\min}$, we have to take $\Delta = \Delta_{\text{C}} = 0$ according to (83). Since $a_{\text{in}} = a = 0$ for a squeezed vacuum state, the choice of ω_{L} corresponds to the center of the squeezing broadband. Introducing the parameter $\rho_{\text{at}} = \gamma/\kappa_{\text{a}}$, we deduce from equations (90, 41) that:

$$\Delta S_{\min\ (\Delta=\Delta_{\text{C}}=0)} = 1 + \left(\mathrm{e}^{-2r} - 1\right) \frac{2C}{(2C+1)(1+\rho_{\text{at}})} \cdot \tag{96}$$

C is the cooperativity parameter characterizing the strength of the coupling between atoms and field:

$$C = \frac{N g_{\text{at}}^2}{2\kappa_{\text{a}}\gamma} \cdot \tag{97}$$

For a large number of atoms ($C \gg 1$) we obtain:

$$\Delta S_{\min(\Delta=\Delta_{\text{C}}=0,\, C\gg 1)} = 1 + \frac{1}{1+\rho_{\text{at}}}(\mathrm{e}^{-2r} - 1). \tag{98}$$

This equality is the equivalent of the one obtained in (91) for the harmonic oscillator. It shows that the spin squeezing is deteriorated by the coupling of the atoms with the vacuum field the rate of which is γ. But the spin noise can be made very small if γ is much smaller than κ_{a}, which characterizes the coupling of the atoms with the cavity mode: significant squeezing transfer corresponds to $\rho_{\text{at}} \ll 1$.

5 Results for the squeezing transfer

We discuss the noise reduction in the case of the harmonic oscillator, which is valid for any value of the squeezing rate of the incoming field. When $\Delta_{\text{b}} = \Delta_{\text{C}} = 0$, the noise spectrum of the minimal quadrature of the harmonic oscillator is obtained from equation (86). If η is large enough, the noise spectrum exhibits two peaks, occurring at two opposite frequencies $\pm\omega_{\text{c}}$. When $\eta \gg 1, \rho$, 0we have $\omega_{\text{c}} \approx g$. In Figure 2 we have plotted the noise spectra: Figure 2a shows the noise spectrum of any quadrature of the harmonic oscillator in the coherent state ($r = 0$), and Figure 2b the noise spectrum of the minimal quadrature when the amount of squeezing of the incoming field is 80% ($\mathrm{e}^{-2r} = 0.2$). The value of ρ is 0.1 and η is equal to 10, which corresponds to $\omega_{\text{c}}/\kappa_{\text{a}} = \Omega_{\text{c}} = 10$.

In Figure 3 we show the evolution of the minimal variance of the harmonic oscillator with $R_{A_{\text{in}}}$ the squeezing rate of the incoming field ($R_{A_{\text{in}}} = 1 - \mathrm{e}^{-2r}$) for several values of η in the squeezing transfer regime ($\Delta_{\text{C}} = \Delta_{\text{b}} = 0$).

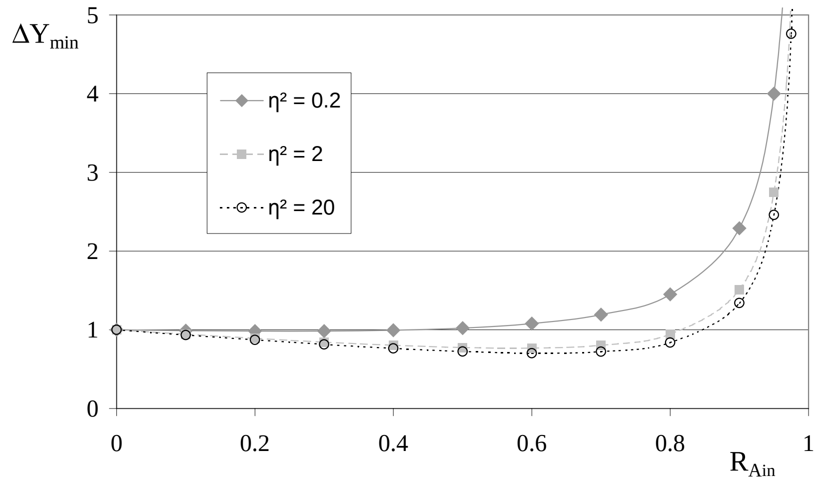

Fig. 4. Evolution of the minimal variance $\Delta Y_{\min}$ with the squeezing rate of the incoming field $R_{A_{\text{in}}}$ for different values of the coupling constant η when the conditions for optimal squeezing transfer are not fulfilled ($\Delta_{\text{C}} = 0$ and $\Delta_{\text{b}} = 1$).

These curves are plotted for $\rho = 0.1$. The limit 1/11, corresponding to perfect field squeezing and high value of η, is given by equation (92).

In Figure 4 the conditions for optimal squeezing transfer are not fulfilled: we have $\Delta_{\text{C}} = 0$ and $\Delta_{\text{b}} = 1$. We see that the squeezing decreases as the field squeezing becomes large, and that the excess noise goes to infinity for perfect squeezed field: when the integral I_{c} is smaller than integral I_{a}, high values of field squeezing degrade the squeezing of the harmonic oscillator. The minimal quadrature gets a contribution from the excess noise quadrature of the field, and consequently its variance tends to infinity when the field is perfectly squeezed.

6 Fluctuations of the reflected field

We now compute the noise spectra of the field reflected by the cavity when an incoming squeezed field interacts with a harmonic oscillator. We will concentrate on the case of the optimal squeezing transfer given by (83). In the same way as we have calculated the fluctuations of the harmonic oscillator in Section 2, we break down the fluctuations of the reflected field into the fluctuations of the incoming field and of the intracavity bath, because these fluctuations are uncorrelated. From equation (46), one deduces the expression of the intracavity field fluctuations:

$$\delta A(\omega) = \frac{\mathrm{i}g}{-\mathrm{i}\omega + \kappa_{\text{a}} + \mathrm{i}\Delta_{\text{C}}}\delta b(\omega) + \frac{\sqrt{2\kappa_{\text{a}}}}{-\mathrm{i}\omega + \kappa_{\text{a}} + \mathrm{i}\Delta_{\text{C}}}\delta A_{\text{in}}(\omega). \quad (99)$$

Using equations (33, 50) we easily obtain the reflected field fluctuations $\delta A_{\text{out}}(\omega)$ for $\Delta_{\text{C}} = \Delta_{\text{b}} = 0$:

$$\delta A_{\text{out}}(\omega)_{\Delta_{\text{C}}=\Delta_{\text{b}}=0} = K_{\text{a}}(\omega)\delta A_{\text{in}}(\omega) + K_{\text{b}}(\omega)\delta b_{\text{in}}(\omega) \quad (100)$$

with:

$$K_{\text{a}}(\omega) = \frac{-g^2 + (\mathrm{i}\omega + \kappa_{\text{a}})(-\mathrm{i}\omega + \kappa_{\text{b}})}{g^2 + (-\mathrm{i}\omega + \kappa_{\text{a}})(-\mathrm{i}\omega + \kappa_{\text{b}})} \quad (101)$$

$$K_{\text{b}}(\omega) = \frac{2\mathrm{i}g\sqrt{\kappa_{\text{a}}\kappa_{\text{b}}}}{g^2 + (-\mathrm{i}\omega + \kappa_{\text{a}})(-\mathrm{i}\omega + \kappa_{\text{b}})}\,\cdot \quad (102)$$

$\delta A^{\dagger}_{\text{out}}(\omega)$ is given by the Hermitian conjugate of this equation, and we thus obtain the following matrix form equality:

$$|\delta A_{\text{out}}(\omega)]_{\Delta_{\text{C}}=\Delta_{\text{b}}=0} = [K_{\text{a}}(\omega)]\,|\delta A_{\text{in}}(\omega)] + [K_{\text{b}}(\omega)]\,|\delta b_{\text{in}}(\omega)] \quad (103)$$

with:

$$|\delta A_{\text{out}}(\omega)] = \begin{vmatrix} \delta A_{\text{out}}(\omega) \\ \delta A^{\dagger}_{\text{out}}(\omega) \end{bmatrix},\ [K_{\text{a}}(\omega)] = \begin{bmatrix} K_{\text{a}}(\omega) & 0 \\ 0 & K_{\text{a}}(\omega) \end{bmatrix},$$

$$[K_{\text{b}}(\omega)] = \begin{bmatrix} K_{\text{b}}(\omega) & 0 \\ 0 & -K_{\text{b}}(\omega) \end{bmatrix}. \quad (104)$$

From equation (103) we deduce the value of the covariance matrix of the reflected field:

$$[V_{A_{\text{out}}}(\omega)]_{\Delta_{\text{C}}=\Delta_{\text{b}}=0} = |K_{\text{a}}(\omega)|^2\,[V_{A_{\text{in}}}(\omega)] + |K_{\text{b}}(\omega)|^2\,[V_{b_{\text{in}}}(\omega)]\,. \quad (105)$$

For the value of the covariance matrix of the squeezed field given by equation (62), we finally get:

$$[V_{A_{\text{out}}}(\omega)]_{\Delta_{\text{C}}=\Delta_{\text{b}}=0} = \begin{bmatrix} |K_{\text{a}}(\omega)|^2\cosh^2(r) + |K_{\text{b}}(\omega)|^2 & |K_{\text{a}}(\omega)|^2\sinh(2r)/2 \\ |K_{\text{a}}(\omega)|^2\sinh(2r)/2 & |K_{\text{a}}(\omega)|^2\sinh^2(r) \end{bmatrix}. \quad (106)$$

The quadrature components $\hat{E}_{\text{out},\varphi}$ of the reflected field are defined in the same way as the ones of the incoming field (see Eq. (63)). From equation (106) we see that the quadrature component corresponding to $\varphi = \pi/2$ has the minimal noise spectrum for all values of the noise frequency, given by:

$$S_{\hat{E}_{\mathrm{out},\varphi=\pi/2}}(\omega)_{\Delta_\mathrm{C}=\Delta_\mathrm{b}=0} = |K_\mathrm{a}(\omega)|^2\,\mathrm{e}^{-2r} + |K_\mathrm{b}(\omega)|^2\,. \tag{107}$$

The quadrature component corresponding to $\varphi = 0$ has the maximal noise spectrum for all frequencies:

$$S_{\hat{E}_{\mathrm{out},\varphi=0}}(\omega)_{\Delta_\mathrm{C}=\Delta_\mathrm{b}=0} = |K_\mathrm{a}(\omega)|^2\,\mathrm{e}^{2r} + |K_\mathrm{b}(\omega)|^2\,. \tag{108}$$

We note that we have the following equality:

$$|K_\mathrm{a}(\omega)|^2 + |K_\mathrm{b}(\omega)|^2 = 1. \tag{109}$$

These results are also valid for the case of a field going out of a cavity containing atoms when the incoming field is in a squeezed vacuum state.

From equations (107–109) we deduce that if the incoming field is in a coherent state ($r = 0$), the noise spectra of all the quadratures of the reflected field, which are the same, are equal to 1 for any value of the analysis frequency ω: the reflected field is in a coherent state as well. In Figure 5 we show for an incoming squeezed field ($\mathrm{e}^{-2r} = 0.2$) the minimal (Fig. 5a) and maximal (Fig. 5b) noise spectra of the reflected field for $\rho = 0.1$ and $\eta = 10$ in the case of optimal squeezing transfer, $\Delta_\mathrm{C} = \Delta_\mathrm{b} = 0$. We see that these spectra have the same constant values as the ones of the incoming field, equal respectively to 0.2 and 5, except near the two coupling frequencies $\pm\Omega_\mathrm{c}$. Around these frequencies the field fluctuations enter the cavity and squeezing or excess noise are transferred in part to the harmonic oscillator. The squeezed quadrature goes out with additional noise, while the antisqueezed quadrature goes out with reduced fluctuations.

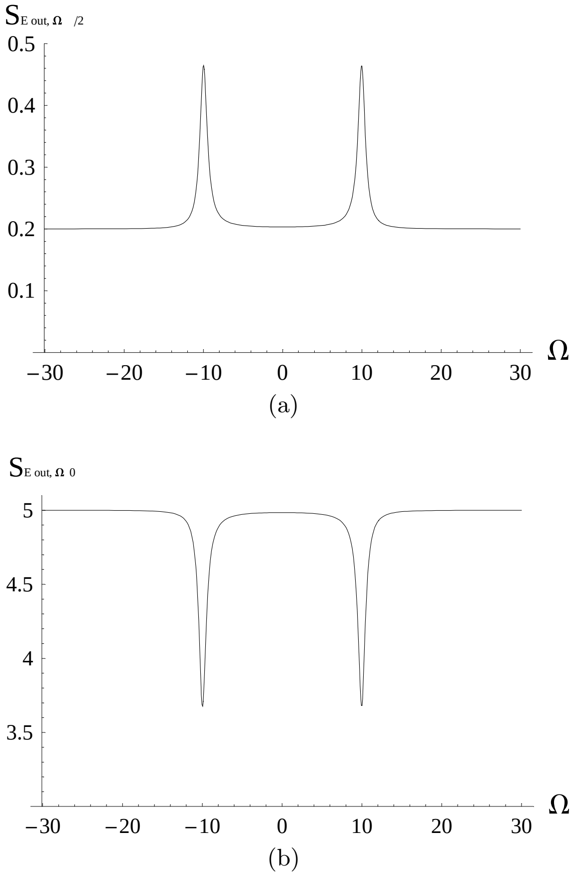

Fig. 5. Noise spectra of the extremal quadrature components of the reflected field in the optimal squeezing transfer regime, with $\rho = 0.1$, $\eta = 10$, $\mathrm{e}^{-2r} = 0.2$. The minimal noise spectrum is plotted in (a) while the maximal noise spectrum is plotted in (b).

7 Correlations between atoms and incoming field

In this part we first show that EPR-type correlations exist between the quadratures of an incoming squeezed field and the quadratures of the harmonic oscillator with which the field interacts. We will be interested in the extremal quadratures. We have seen in Section 3 that the minimal noise quadratures are $\hat{E}_{\mathrm{in},\,\varphi=\frac{\pi}{2}}$ and $Y_{\varphi=0}$, while the maximal noise quadratures are $\hat{E}_{\mathrm{in},\,\varphi=0}$ and $Y_{\varphi=\frac{\pi}{2}}$. Using equations (50, 51), and the equalities $J_\mathrm{a}(\omega) = -J_\mathrm{a}(-\omega)^*$, $J_\mathrm{b}(\omega) = J_\mathrm{b}(-\omega)^*$ when $\Delta_\mathrm{C} = \Delta_\mathrm{b} = 0$, one obtains easily the following relations:

$$\delta Y_{\varphi=0}(\omega) = -\mathrm{i}J_\mathrm{a}(\omega)\delta\hat{E}_{\mathrm{in},\,\varphi=\frac{\pi}{2}}(\omega) + J_\mathrm{b}(\omega)(\delta b_\mathrm{in}(\omega) + \delta b_\mathrm{in}^\dagger(\omega)) \tag{110}$$

$$\delta Y_{\varphi=\frac{\pi}{2}}(\omega) = \mathrm{i}J_\mathrm{a}(\omega)\delta\hat{E}_{\mathrm{in},\,\varphi=0}(\omega) + \mathrm{i}J_\mathrm{b}(\omega)(\delta b_\mathrm{in}(\omega) - \delta b_\mathrm{in}^\dagger(\omega)). \tag{111}$$

We define the normalized correlation function between two operators O and O', the noise spectra of which are respectively $S_O(\omega)$ and $S_{O'}(\omega)$, from the noise spectra of the sum $O + O'$, $S_{O+O'}(\omega)$:

$$C_{O,O'}(\omega) = \frac{S_{O+O'}(\omega) - S_O(\omega) - S_{O'}(\omega)}{2\sqrt{S_O(\omega)S_{O'}(\omega)}}\,\cdot \tag{112}$$

By calculating $S_{O+O'}(\omega)$ we show that $C_{O,O'}(\omega)$ is related to the statistical average of $\delta O(\omega)\delta O'(\omega')$ by the relation

$$\mathrm{Re}(\langle\delta O(\omega)\delta O'(\omega')\rangle) = 2\pi\delta(\omega+\omega')\sqrt{S_O(\omega)S_{O'}(\omega)}\;C_{O,O'}(\omega) \tag{113}$$

and obeys the inequality: $-1 \le C_{O,O'}(\omega) \le 1$, with perfect correlation when $C_{O,O'}(\omega) = 1$, perfect anticorrelation when $C_{O,O'}(\omega) = -1$, and no correlation for $C_{O,O'}(\omega) = 0$.

Since incoming field and harmonic oscillator bath are uncorrelated, we derive from equations (110, 111) the four correlation functions:

$$C_{0,\frac{\pi}{2}}(\Omega) = C_{Y_{\varphi=0},\hat{E}_{\mathrm{in},\,\varphi=\frac{\pi}{2}}}(\Omega) = \frac{\mathrm{Im}(J_\mathrm{a}(\omega))}{\sqrt{|J_\mathrm{a}(\omega)|^2 + \mathrm{e}^{2r}\,|J_\mathrm{b}(\omega)|^2}}$$
$$= \frac{\eta^2+\rho-\Omega^2}{|D(\Omega)|}\frac{1}{\sqrt{1+\frac{\rho}{\eta^2}\mathrm{e}^{2r}(1+\Omega^2)}} \quad (114)$$

$$C_{\frac{\pi}{2},0}(\Omega) = C_{Y_{\varphi=\frac{\pi}{2}},\hat{E}_{\mathrm{in},\,\varphi=0}}(\Omega) = \frac{-\,\mathrm{Im}(J_\mathrm{a}(\omega))}{\sqrt{|J_\mathrm{a}(\omega)|^2 + \mathrm{e}^{-2r}\,|J_\mathrm{b}(\omega)|^2}}$$
$$= \frac{\eta^2+\rho-\Omega^2}{|D(\Omega)|}\frac{-1}{\sqrt{1+\frac{\rho}{\eta^2}\mathrm{e}^{-2r}(1+\Omega^2)}} \quad (115)$$

$$C_{0,0}(\Omega) = C_{Y_{\varphi=0},\hat{E}_{\mathrm{in},\,\varphi=0}}(\Omega) = \frac{\mathrm{Re}(J_\mathrm{a}(\omega))}{\sqrt{|J_\mathrm{a}(\omega)|^2 + \mathrm{e}^{2r}\,|J_\mathrm{b}(\omega)|^2}}$$
$$= \frac{(\rho+1)\Omega}{|D(\Omega)|}\frac{1}{\sqrt{1+\frac{\rho}{\eta^2}\mathrm{e}^{2r}(1+\Omega^2)}} \quad (116)$$

$$C_{\frac{\pi}{2},\frac{\pi}{2}}(\Omega) = C_{Y_{\varphi=\frac{\pi}{2}},\hat{E}_{\mathrm{in},\,\varphi=\frac{\pi}{2}}}(\Omega) = \frac{-\,\mathrm{Re}(J_\mathrm{a}(\omega))}{\sqrt{|J_\mathrm{a}(\omega)|^2 + \mathrm{e}^{-2r}\,|J_\mathrm{b}(\omega)|^2}}$$
$$= \frac{(\rho+1)\Omega}{|D(\Omega)|}\frac{-1}{\sqrt{1+\frac{\rho}{\eta^2}\mathrm{e}^{-2r}(1+\Omega^2)}} \quad (117)$$

where $\Omega = \omega/\kappa_\mathrm{a}$ and $D(\Omega)$ is given by (54). As previously we obtain the same results for atoms interacting with a squeezed vacuum field.

The four corresponding correlation functions $C_{0,\frac{\pi}{2}}(\Omega)$, $C_{\frac{\pi}{2},0}(\Omega)$, $C_{0,0}(\Omega)$ and $C_{\frac{\pi}{2},\frac{\pi}{2}}(\Omega)$ are plotted in Figure 6 for $\rho = 0.1$, $\eta = 10$ and $\mathrm{e}^{-2r} = 0.2$.

We see that the squeezed quadratures of the atomic system and field, $Y_{\varphi=0}$ and $\hat{E}_{\mathrm{in},\,\varphi=\frac{\pi}{2}}$, are well correlated or anticorrelated outside the resonance peaks at $\Omega = \pm\Omega_\mathrm{c} = \pm 10$. In the vicinity of the resonance peaks, the correlation goes to zero (see Fig. 6a). The same is true for the two antisqueezed quadratures (see Fig. 6b). On the other hand, the correlations either between $Y_{\varphi=0}$ and $\hat{E}_{\mathrm{in},\,\varphi=0}$, or $Y_{\varphi=\pi/2}$ and $\hat{E}_{\mathrm{in},\,\varphi=\pi/2}$, are maximum close to the resonance frequencies and zero for other frequencies (see respectively Fig. 6c and Fig. 6d).

From equations (114–117), we can compute the inferred spectra of the incoming field when the one of the harmonic oscillator is known [16]:

$$S^{\mathrm{inf}}_{\hat{E}_{\mathrm{in},\,\varphi=0}}(\Omega) = S_{\hat{E}_{\mathrm{in},\,\varphi=0}}(1 - C_{i,\,0}(\Omega)^2)$$
$$= \mathrm{e}^{2r}(1 - C_{i,\,0}(\Omega)^2) \quad (118)$$

$$S^{\mathrm{inf}}_{\hat{E}_{\mathrm{in},\,\varphi=\pi/2}}(\Omega) = S_{\hat{E}_{\mathrm{in},\,\varphi=\pi/2}}(1 - C_{j,\,\pi/2}(\Omega)^2)$$
$$= \mathrm{e}^{-2r}(1 - C_{j,\,\pi/2}(\Omega)^2) \quad (119)$$

where we can use either ($i = 0$, $j = \pi/2$) or ($i = \pi/2$, $j = 0$) to infer the field spectra.

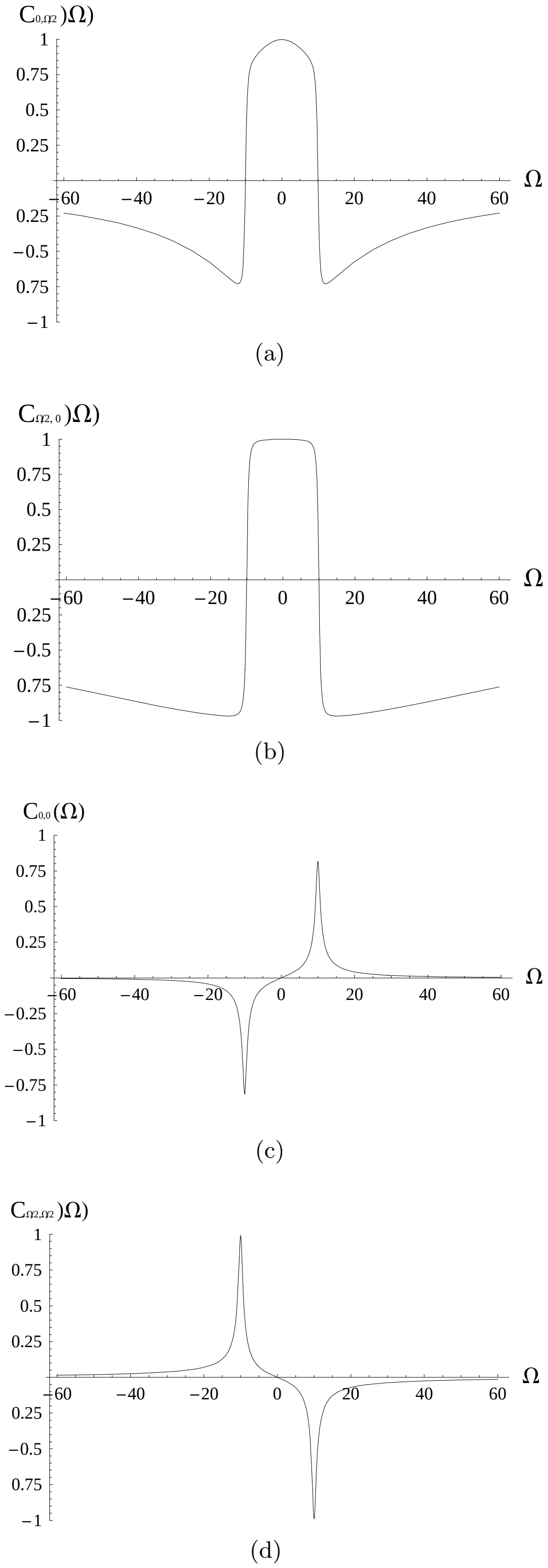

Fig. 6. The four correlation functions between the squeezed quadrature and antisqueezed quadrature of the harmonic oscillator and squeezed and antisqueezed quadratures of the incoming field for $\Delta_\mathrm{C} = \Delta_\mathrm{b} = 0$, $\rho = 0.1$, $\eta = 10$, and 80% of squeezing of the incoming field.

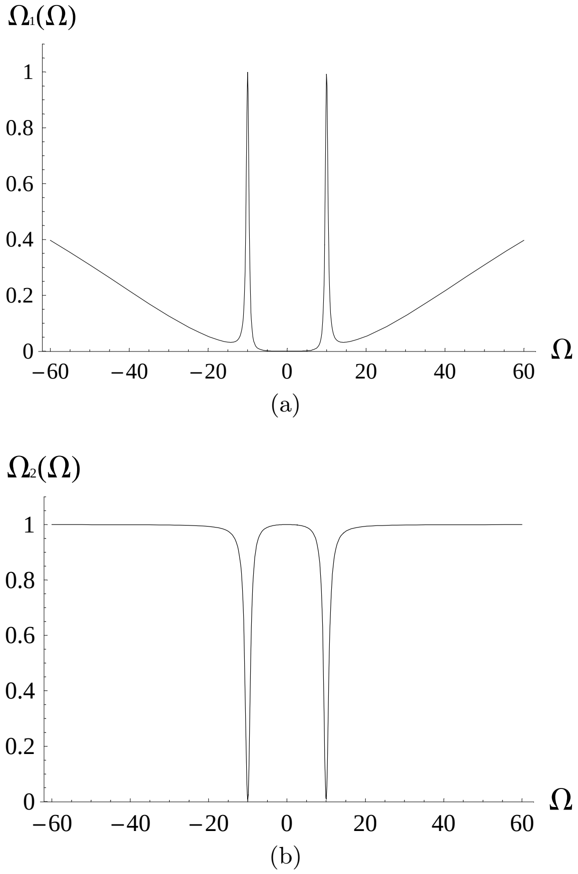

Fig. 7. Products of the inferred spectra of the incoming field quadratures *versus* the normalized frequency Ω for $\Delta_C = \Delta_b = 0$, $\rho = 0.1$ and $\eta = 10$. For $\Pi_1(\Omega)$ plotted in (a) the squeezed and antisqueezed quadratures of the incoming field are respectively inferred from the squeezed and antisqueezed quadrature of the harmonic oscillator. For $\Pi_2(\Omega)$ plotted in (b) they are respectively inferred from the antisqueezed and squeezed quadrature.

Figure 7 shows for $\rho = 0.1$, $\eta = 10$ and $e^{-2r} = 0.2$ the products $\Pi_1(\Omega)$ and $\Pi_2(\Omega)$, with:

$$\begin{aligned}\Pi_1(\Omega) &= S^{\text{inf}}_{\hat{E}_{\text{in}},\,\varphi=0}(\Omega)S^{\text{inf}}_{\hat{E}_{\text{in}},\,\varphi=\pi/2}(\Omega)_{(i=\pi/2,\,j=0)}\\ &= (1 - C_{\pi/2,0}(\Omega)^2)(1 - C_{0,\pi/2}(\Omega)^2) \qquad (120)\end{aligned}$$

$$\begin{aligned}\Pi_2(\Omega) &= S^{\text{inf}}_{\hat{E}_{\text{in}},\,\varphi=0}(\Omega)S^{\text{inf}}_{\hat{E}_{\text{in}},\,\varphi=\pi/2}(\Omega)_{(i=0,\,j=\pi/2)}\\ &= (1 - C_{0,0}(\Omega)^2)(1 - C_{\pi/2,\,\pi/2}(\Omega)^2). \qquad (121)\end{aligned}$$

It can be seen that $\Pi_1(\Omega)$ goes below one around $\Omega = 0$ and for frequencies larger than Ω_c or smaller than $-\Omega_c$, while $\Pi_2(\Omega)$ goes below one around $\Omega = \pm\Omega_c$. This shows that measurements on two conjugate quadratures of the harmonic oscillator allows to infer the incoming field quadratures with accuracies apparently violating Heisenberg inequalities which corresponds to EPR-type correlations.

8 Conclusion

We have shown that an ensemble of two level atoms driven by a squeezed vacuum field with a low photon number (*i.e.* moderate squeezing) is equivalent to a harmonic oscillator, and consequently can be treated fully analytically. The squeezing transfer from the field to the atoms can be very good at the condition that the coupling of the atomic system to its reservoir is very small, and the coupling between the field and the atomic system is large ($g^2 \gg \kappa_a\kappa_b$, and $\kappa_a > \kappa_b$). Strong coupling as defined as $g \gg \kappa_a, \kappa_b$ is however not necessary. Once the squeezed driving field id turned off, the lifetime of the atomic squeezing is of the order of the lifetime of the atomic dipole in the cavity which is short for an atomic transition. Strong EPR-type correlations are found between the quadrature components of the harmonic oscillator and the field. The considered two level atom or harmonic oscillator is a model system that has allowed us to highlight the main conditions for squeezing transfer. Here the squeezing imprinted on the atoms can only be tested indirectly through the measurements on the outgoing field.

Atomic three level systems interacting with a squeezed field and a coherent field have already been considered in the case of single pass interaction. They allow independent probing of long lived atomic coherences that are likely to be squeezed [5]. It is possible to generalize our results to a three level system interacting with two fields in a cavity. This will be the subject of a forthcoming publication.

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